Improving the Locality of Generalized Integrated Interleaved Code

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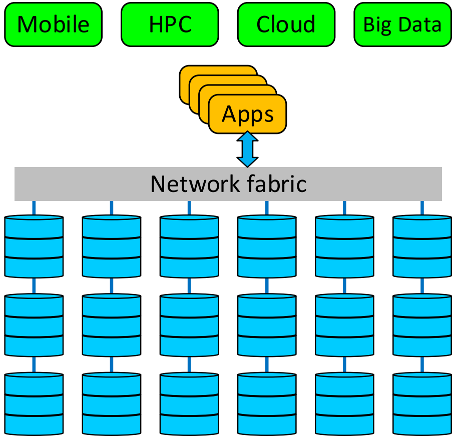
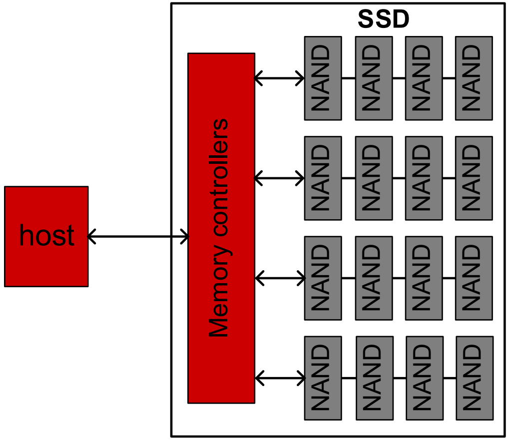
Application of Generalized Integrated Interleaved (GII) Codes





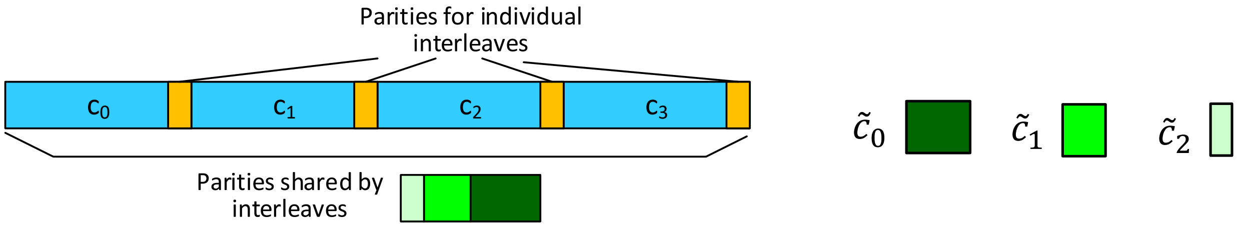
- Redundancy is needed to tolerate and recover from failures in distributed storage
- GII codes are a type of locally recoverable erasure codes that substantially reduce the latency and overhead of failure recovery

- Enable short sub-codewords to be adopted for error correction without substantial redundancy increase
- Achieve >40GBytes/sec throughput with excellent correction capability

Outline

- Generalized integrated interleaved (GII) codes
- Modified two-layer GII codes with improved locality
- A generalization of three-layer integrated interleaved codes
- Conclusions

Generalized Integrated Interleaved (GII) Codes



$$\begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \vdots \\ \tilde{c}_{v-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(v-1)} & \alpha^{2(v-1)} & \cdots & \alpha^{(v-1)(m-1)} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{m-1} \end{bmatrix}$$

Nesting matrix G

Correction capability $t_v \geq t_{v-1} \geq \cdots \geq t_1 \geq t_0$

$$\begin{array}{ccccccccc} \mathcal{C}_v & \subseteq & \mathcal{C}_{v-1} & \subseteq & \cdots & \subseteq & \mathcal{C}_1 & \subseteq & \mathcal{C}_0 \\ \Cup & & \Cup & & & & \Cup & & \Cup \\ \tilde{c}_0 & & \tilde{c}_1 & & & & \tilde{c}_{v-1} & & c_0, c_1, \cdots, c_{m-1} \end{array}$$

Decoding of GII Codes

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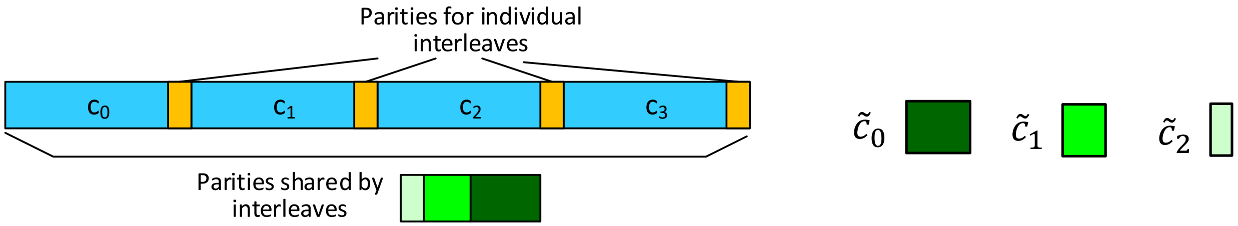

Interleave syndromes — nested syndromes

$$\begin{bmatrix} S_j^{(l_1)} \\ S_j^{(l_2)} \\ \vdots \\ S_j^{(l_b)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha^{l_1} & \alpha^{l_2} & \cdots & \alpha^{l_b} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha^{(b-1)l_1} & \alpha^{(b-1)l_2} & \cdots & \alpha^{(b-1)l_b} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{S}_j^{(0)} \\ \tilde{S}_j^{(1)} \\ \vdots \\ \tilde{S}_j^{(b-1)} \end{bmatrix}$$

Syndrome conversion matrix

$l_1, l_2, \cdots, l_b$: indices of interleaves with more than t_0 erasures (exceptional interleaves)

- t syndromes are needed to correct t erasures
- Higher-order syndromes for the interleaves are generated from the nested syndromes
- Syndrome conversion matrix is always invertible
- Each nested syndrome is generated by utilizing all interleaves

Modified GII Codes

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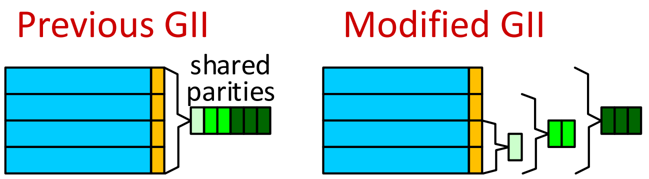
$$G' = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \alpha & \cdots & \alpha^{v-2} & \alpha^{v-1} & \cdots & \alpha^{m-1} \\ 0 & 0 & \cdots & \alpha^{2(v-2)} & \alpha^{2(v-1)} & \cdots & \alpha^{2(m-1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha^{(v-1)^2} & \cdots & \alpha^{(v-1)(m-1)} \end{bmatrix}$$



- Less powerful nestings involve fewer interleaves
- Form the syndrome conversion matrix using the bottom rows of G' as much as possible
  - The selected nestings should have sufficient correction capability
  - The selected nestings should cover the exceptional interleaves
  - Consecutive nestings are used to simplify the selection
- The syndrome conversion matrix is invertible if the number of interleaves does not exceed the values in the table

v	$GF(2^4)$	$GF(2^5)$	$GF(2^6)$	$GF(2^7)$	$GF(2^8)$	$GF(2^9)$
2	16	32	64	128	256	512
3	5	6	12	22	28	62
4	5	6	8	12	15	20

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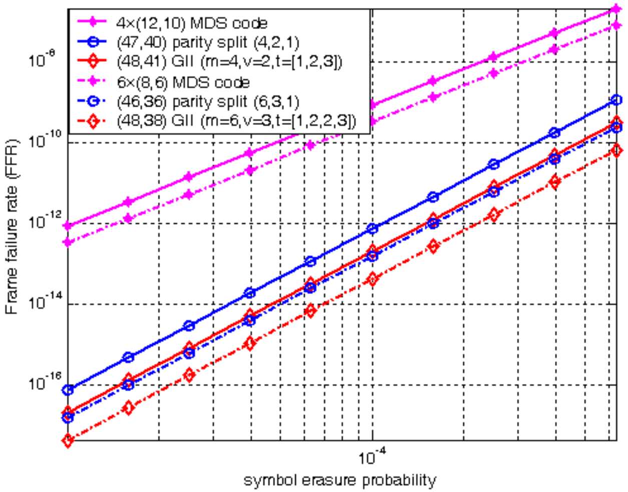
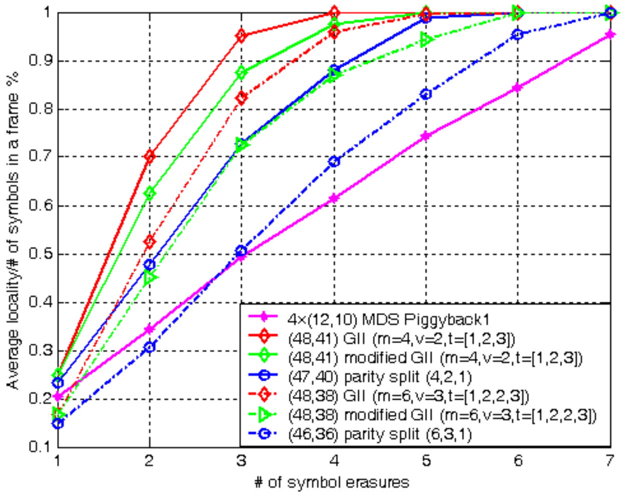
Correction Capability and Locality Comparisons





- Modified GII codes preserve the same correction capability as the original GII codes for most practical settings
- Modified GII codes require fewer interleaves to utilize the shared parities when there are fewer extra erasures to correct
- Have very small implementation overhead compared to the GII codes
- Achieve good tradeoff on locality and correction capability

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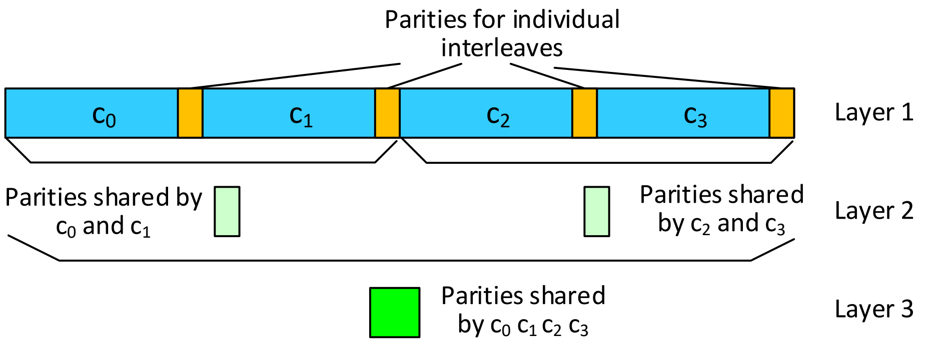
Three-layer Integrated Interleaved Codes



- Layer 2 parities are shared by individual subgroups of interleaves
- Layer 3 parities are shared by all interleaves
- When there are fewer interleaves with fewer extra errors/erasures to correct, layer 2 parities and the other interleaves in the same sub-group are utilized

Nestings in Three-Layer Codes

G_1: layer-2 nesting matrix

G_2: layer-3 nesting matrix

$$\Gamma = \left[\begin{array}{c|c|c|c} G_1 & 0 & \cdots & 0 \\ \hline 0 & G_1 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & G_1 \\ \hline \multicolumn{4}{c}{G_2} \end{array}\right] = \left[\begin{array}{c|c|c|c} 11\cdots1 & 0 & \cdots & 0 \\ \hline 0 & 11\cdots1 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & 11\cdots1 \\ \hline 11\cdots1 & 11\cdots1 & 11\cdots1 & 11\cdots1 \end{array}\right]$$

Joint nesting matrix

Previous 3-layer nesting

- Previous 3-layer integrated interleaved code
  - One-level of nesting in each layer
  - Only one exceptional interleave from each subgroup can be corrected
  - Layer-3 parities only add to the correction of a single exceptional interleave in a subgroup

Generalized Three-layer Integrated Interleaved Codes

$$\Gamma = \begin{bmatrix} \begin{array}{c|c|c|c} G_1 & 0 & \cdots & 0 \\ \hline 0 & G_1 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & G_1 \\ \hline \multicolumn{4}{c}{G_2} \end{array} \end{bmatrix} = \begin{bmatrix} \begin{array}{c|c|c|c} G_1 & 0 & \cdots & 0 \\ 0 & G_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_1 \\ \hline G_2^{r-1} & G_2^{r-1} & \cdots & G_2^{r-1} \\ G_2^{r-2} & \alpha^{-1} G_2^{r-2} & \cdots & \alpha^{-(m_1-1)} G_2^{r-2} \\ \vdots & \vdots & \ddots & \vdots \\ G_2^{0} & \alpha^{-(r-1)} G_2^{0} & \cdots & \alpha^{-(m_1-1)(r-1)} G_2^{0} \end{array} \end{bmatrix}$$

v_1: # of levels of codes in layer 2

m_1: # of subgroups

m_2: # of interleaves in a subgroup

r: # of subgroups with extra interleaves to be corrected

s_i: # of extra interleaves in a subgroup to be corrected

$$G_1 = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{m_2-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{(v_1-1)} & \alpha^{2(v_1-1)} & \cdots & \alpha^{(v_1-1)(m_2-1)} \end{bmatrix}$$

$$G_2^i = \begin{bmatrix} 1 & \alpha^{v_1} & \alpha^{2v_1} & \cdots & \alpha^{v_1(m_2-1)} \\ 1 & \alpha^{v_1+1} & \alpha^{2(v_1+1)} & \cdots & \alpha^{(v_1+1)(m_2-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{v_1+s_i-1} & \alpha^{2(v_1+s_i-1)} & \cdots & \alpha^{(v_1+s_i-1)(m_2-1)} \end{bmatrix}$$

- The syndrome conversion matrix formed by the columns corresponding to the interleaves with extra erasures and consecutive rows of $[G_1^T | (G_2^i)^T]^T$ in the joint nesting matrix is always invertible, if the numbers of groups and exceptional interleaves in the groups are not exceeded

Correctable Erasure Pattern Comparisons

Example 1: same redundancy
single-level nesting in 3-layer GII code

Layer-2: $t_1 = t_0 + 1$; layer-3: $t'_1 = t_0 + 2$

$$\Gamma_{II} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \Gamma_{GII} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & \alpha & 1 & \alpha \end{bmatrix}$$

Pattern	GII	II	Pattern	GII	II
$\{1,0 \mid 0,0\}$	✓	✓	$\{1,0 \mid 1,0\}$	✓	✓
$\{1,1 \mid 0,0\}$	✓	✗	$\{1,1 \mid 1,0\}$	✓	✗
$\{2,0 \mid 0,0\}$	✓	✓	$\{2,0 \mid 1,0\}$	✓	✓
$\{2,1 \mid 0,0\}$	✓	✗	$\{2,1 \mid 1,0\}$	✓	✗

- Multiple code levels are allowed in layer 2 and 3
- Layer-3 codes can correct additional interleaves
- Layer-3 codes do not have to be stronger than layer-2 codes

Example 2: same redundancy
multi-level nesting in 3-layer GII code

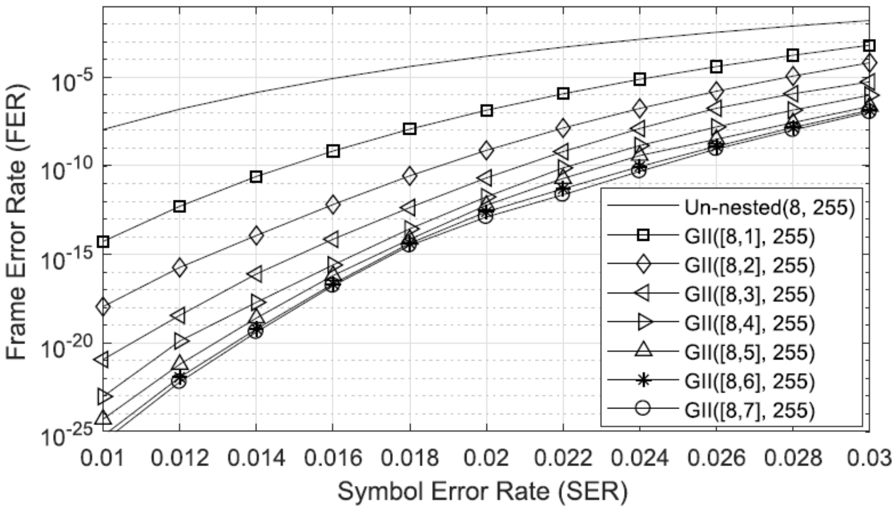
II codes: $t_1 = t_0 + 2$; $t'_1 = t_0 + 3$

GII codes: $t_1 = t_2 = t_0 + 1$; layer-3: $t'_1 = t_0 + 1$; $t'_2 = t_0 + 2$

Extra erasure patterns correctable by the 3-layer II code but not the 3-layer GII code	$\{2,0,0 \mid 2,0,0 \mid 0,0,0\}\{2,0,0 \mid 2,0,0 \mid 1,0,0\}$ $\{2,0,0 \mid 2,0,0 \mid 2,0,0\}$
	$\{3,0,0 \mid 0,0,0 \mid 0,0,0\}\{3,0,0 \mid 1,0,0 \mid 0,0,0\}$ $\{3,0,0 \mid 1,0,0 \mid 1,0,0\}\{3,0,0 \mid 2,0,0 \mid 0,0,0\}$ $\{3,0,0 \mid 2,0,0 \mid 1,0,0\}\{3,0,0 \mid 2,0,0 \mid 2,0,0\}$
Extra erasure patterns correctable by the 3-layer GII code but not the 3-layer II code	$\{1,1,0 \mid 0,0,0 \mid 0,0,0\}\{1,1,0 \mid 1,0,0 \mid 0,0,0\}$ $\{1,1,0 \mid 1,0,0 \mid 1,0,0\}\{1,1,0 \mid 1,1,0 \mid 0,0,0\}$ $\{1,1,0 \mid 1,1,0 \mid 1,0,0\}\{1,1,0 \mid 1,1,0 \mid 1,1,0\}$
	$\{1,1,1 \mid 0,0,0 \mid 0,0,0\}\{1,1,0 \mid 1,0,0 \mid 0,0,0\}$ $\{1,1,1 \mid 1,0,0 \mid 1,0,0\}\{1,1,0 \mid 1,1,0 \mid 0,0,0\}$ $\{1,1,1 \mid 1,1,0 \mid 1,0,0\}\{1,1,0 \mid 1,1,0 \mid 1,1,0\}$
	$\{2,1,0 \mid 0,0,0 \mid 0,0,0\}\{2,1,0 \mid 1,0,0 \mid 0,0,0\}$ $\{2,1,0 \mid 1,0,0 \mid 1,0,0\}\{2,1,0 \mid 1,1,0 \mid 0,0,0\}$ $\{2,1,0 \mid 1,1,0 \mid 1,0,0\}\{2,1,0 \mid 1,1,0 \mid 1,1,0\}$
	$\{2,1,1 \mid 0,0,0 \mid 0,0,0\}\{2,1,1 \mid 1,0,0 \mid 0,0,0\}$ $\{2,1,1 \mid 1,0,0 \mid 1,0,0\}\{2,1,1 \mid 1,1,0 \mid 0,0,0\}$ $\{2,1,1 \mid 1,1,0 \mid 1,0,0\}\{2,1,1 \mid 1,1,0 \mid 1,1,0\}$
	$\{2,1,1 \mid 1,1,1 \mid 0,0,0\}\{2,1,1 \mid 1,1,1 \mid 1,0,0\}$ $\{2,1,1 \mid 1,1,1 \mid 1,1,0\}$

Implementation Architectures for GII Decoders

- Two-layer GII codes based on Reed-Solomon codes over $GF(2^8)$
  - 8 sub-codewords
  - Length of each sub-codeword: 2040 bit
  - Redundancy: 12.5%



- Hardware complexity
  - <30% area increase compared to hard-decision Reed-Solomon decoder
  - Clock frequency: >550Mhz on FPGA
  - >40GByte/s throughput

* X. Zhang and Z. Xie, "Efficient Architectures for Generalized Integrated Interleaved Decoder," IEEE Trans. on Circuits and Systems-I, 2019.

Conclusions

- GII codes can achieve hyper throughput with excellent correction capability and low complexity
- Modified two-layer GII codes improve the locality of erasure correction without degradation on the correction capability for most practical settings
- Three-layer GII codes further improve the locality of erasure and error correction and the third-layer parities can be used in a flexible way
- Three-layer GII codes achieve better locality than two-layer GII codes at the cost of higher redundancy